

M.U.
M.Sc. 94

Q No \rightarrow Let N and N^* be normed linear spaces and T is a linear transformation of N into N^* . Then the following statements are equivalent:-

- (i) T is continuous,
- (ii) T is continuous at the origin.

Prove this.

or, Q No \rightarrow Let N and N^* be normed linear space and T a linear transformation of N into N^* . Then the following conditions are equivalent to one another:-

- (i) T is continuous,
- (ii) T is continuous at the origin,
[i.e., $x_n \rightarrow 0 \Rightarrow T(x_n) \rightarrow 0$]

(iii) T is bounded.

[i.e., \exists a real number $K \geq 0$ s.t.

$$\|T(x)\| \leq K \|x\| \quad \forall x \in N]$$

(iv) If $S = \{x; \|x\| = 1\}$ is closed sphere in N , then its image $T(S)$ is bounded set N^* .

Proof:- ~~First~~ (i) \Rightarrow (ii)

Let $T: N \rightarrow N^*$ be a continuous map and $\langle x_n \rangle$ be a sequence in N such that

$$\lim_{n \rightarrow \infty} x_n = 0$$

By continuity of T ,

$$x_n \rightarrow 0 \Rightarrow T(x_n) \rightarrow T(0). \text{ But } T(0) = 0 \text{ for any}$$

linear map.

$$\Rightarrow T(x) \rightarrow 0.$$

This Proves that T is Continuous at origin.

Now, we claim (ii) \Rightarrow (i)

Let T be Continuous at the origin.

Let $\langle x_n \rangle$ be a sequence in N such that,

$$\lim_{n \rightarrow \infty} x_n = x \in N.$$

$$\text{Then, } x_n \rightarrow x \Rightarrow x_n - x \rightarrow 0 \Rightarrow T(x_n - x) \rightarrow T(0) = 0.$$

[For T is Continuous at origin].

$$\Rightarrow T(x_n) - T(x) \rightarrow 0$$

$$\Rightarrow T(x_n) \rightarrow T(x).$$

$$\text{Thus, } x_n \rightarrow x \Rightarrow T(x_n) \rightarrow T(x).$$

By defⁿ this Proves that T is Continuous. \checkmark

(II) To Prove (ii) \Rightarrow (iii). Let T ~~is~~ be Continuous at $x=0$. To Prove (iii), Then T is not bounded.

Then for each +ve integer n , we can find vector $x_n \in N$ such that,

$$\|T(x_n)\| > n \|x_n\|$$

$$\text{or, } \frac{\|T(x_n)\|}{n \|x_n\|} > 1. \quad \text{--- (1)}$$

$$\text{or, } \left\| T \left[\frac{x_n}{n \|x_n\|} \right] \right\| = \frac{\|T(x_n)\|}{n \|x_n\|} > 1.$$

$$\text{Taking } y_n = \frac{x_n}{n \|x_n\|}, \text{ we get } T(y_n) > 1 \quad \text{--- (2)}$$

$$\text{Also, } \|y_m\| = \left\| \frac{x_m}{m \|x_m\|} \right\| = \frac{\|x_m\|}{m \|x_m\|} = \frac{1}{m}.$$

$$\therefore \lim_{n \rightarrow \infty} \|y_m\| = \frac{0}{\infty} \quad \text{--- (1)}$$

$$\therefore \|y_m\| = 0 \quad \text{--- (3)}$$

$$\therefore \lim_{n \rightarrow \infty} y_m = 0. \text{ But } \lim_{n \rightarrow \infty} T(y_m) > 1, \text{ by (2)}$$

Finally, $y_m \rightarrow 0 \Rightarrow T(y_m)$ does not tend to zero as $n \rightarrow \infty$.

A ~~Contradiction~~ Contradiction. For T is Continuous at $x=0$.

Hence (iii) is true.

To Prove (iii) \Rightarrow (ii).

Let T be bounded so that \exists a real no. $K \geq 0$.
Such that, $\|T(x)\| \leq K \|x\| \forall x \in N$ --- (4)

Let $\langle x_m \rangle$ be a sequence in N such that,

$$\lim_{n \rightarrow \infty} x_m = 0 \in N.$$

$$\Rightarrow \|x_m\| \rightarrow 0 \text{ as } n \rightarrow \infty \quad \text{--- (5)}$$

For norm is a Continuous map.

$$\text{By (4), } \|T(x_m)\| \leq K \|x_m\|.$$

using (5), we find that $\|T(x_m)\| \rightarrow 0$ as $n \rightarrow \infty$.
 $\Rightarrow T(x_m) \rightarrow 0$ as $n \rightarrow \infty$.

This shows that T is Continuous at the origin.

(iii) To Prove (iii) \Rightarrow (iv)

Let (iii) be true, i.e., let $\exists K > 0$ such that

$$\|T(x)\| \leq K \|x\| \quad \forall x \in N \quad \text{--- (6)}$$

$$\text{(6)} \Rightarrow \|T(x)\| \leq K \|x\| \quad \forall x \in S \subset N \quad \text{--- (7)}$$

But, $x \in S \Rightarrow T(x) \in T(S)$.

Hence, (7) Proves that $T(S)$ is bounded.

To Prove (iv) \Rightarrow (iii). Let (iv) be true, then $\exists K > 0$ such that, $\|T(x)\| \leq K \|x\| \quad \forall x \in S$ --- (8)

If $x=0$, then $T(x)=0$ so, that (8) is true for $x=0$ --- (9).

If $x \neq 0$, then put $y = \frac{x}{\|x\|}$ so that,

$$\|y\| = \frac{\|x\|}{\|x\|} = 1.$$

Consequently $x \in S$ and so by (8) $\cdot \|T(y)\| \leq K \|y\|$

$$\text{or, } \left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq K \left\| \frac{x}{\|x\|} \right\| = \frac{K \|x\|}{\|x\|} = K.$$

$$\text{or, } \|T(x)\| \leq K \quad \forall x \neq 0 \quad \text{--- (10)}$$

From (9) & (10), we have

$$\|T(x)\| \leq K \|x\| \quad \forall x \in N.$$

$\Rightarrow T$ is bounded. Hence (iii)

This completes the Proof.